The Space of Gravitational Degrees of Freedom

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ABSTRACT

In general relativity, physical states are represented by isometry classes of solutions of Einstein's field equations. The resulting space $\mathrm{Grav}(V)$ of the totality of such states on a 4-dimensional manifold V is known as the *space of gravitational degrees of freedom*. We investigate the mathematical structure of this space when V is diffeomorphic to $R \times M$, where M is a closed 3-dimensional manifold which represents the spatial cosmic topology of the universe. One of our main results is how the topology of M gets encoded into the structure of $\mathrm{Grav}(V)$. We give a simple generic topological condition on M such that for spacetimes with constant mean curvature hypersurfaces, $\mathrm{Grav}(V)$ has the natural structure of an infinite-dimensional symplectic manifold on an open dense subset and that on the complement of this set, a nowhere dense subset, $\mathrm{Grav}(V)$ has singularities which are of orbifold type, that is, of a manifold modulo a finite group action.

Let τ be a real number < 0 and let $\mathcal{E}_\tau(V)$ denote the space of *globally-hyperbolic maximally-developed vacuum spacetimes* that admit a Cauchy hypersurface Σ_τ with (CMC) *constant mean curvature* $= \tau$. Let $\mathcal{D}(V)$ denote the group of diffeomorphisms of V. The space of gravitational degrees of freedom of $\mathcal{E}_\tau(V)$ is then defined to be $\mathrm{Grav}_\tau(V) = \mathcal{E}_\tau(V)/\mathcal{D}(V)$. Let $\mathcal{C}_\mathcal{H}(M) \cap \mathcal{C}_\delta(M) \cap \mathcal{C}_\tau(M)$ denote the intersection of the spaces of ADM Cauchy data (g, π) on M that satisfy the Hamiltonian and divergence constraints and that have constant mean curvature τ. Then we construct a natural bijection

$$\mathrm{Grav}_\tau(V) \equiv \frac{\mathcal{E}_\tau(V)}{\mathcal{D}(V)} \overset{\text{bijection}}{\longleftrightarrow} \frac{\mathcal{C}_\mathcal{H}(M) \cap \mathcal{C}_\delta(M) \cap \mathcal{C}_\tau(M)}{\mathcal{D}(M)}, \quad [g_V] \longleftrightarrow [(g_\tau, \pi_\tau)]$$

that maps orbits to orbits and which provides a *natural parameterization* for $\mathrm{Grav}_\tau(V)$.

We give a simple generic topological condition on M, $\deg M = 0$, where $\deg M = \max\{\dim I_g(M) \mid g \in \mathrm{Riem}(M)\}$ is the maximum dimension of all isometry groups $I_g(M)$ of all Riemannian metrics g on M. This condition guarantees that the constraint space $\mathrm{CMC}(\tau) \equiv \mathcal{C}_\mathcal{H}(M) \cap \mathcal{C}_\delta(M) \cap \mathcal{C}_\tau(M)$ is a smooth *global* infinite dimensional manifold of Wheeler dimension $7\infty^3 = (12-1-3-1)\infty^3$. The orbit space of isometry classes of $\mathrm{CMC}(\tau)$ Cauchy data

$(\mathcal{C}_{\mathcal{H}}(M) \cap \mathcal{C}_{\delta}(M) \cap \mathcal{C}_{\tau}(M))/\mathcal{D}(M)$ is then an infinite dimensional manifold of Wheeler dimension $4\infty^3 = (7-3)\infty^3$ away from the orbifold type singularities which occur on a nowhere dense set.

We discuss potential applications of our work to 3-manifold geometrization and cosmology, which if successful, would give a dynamical reason, provided by Einstein's equations, to explain the observed fact that the universe is spatially locally homogeneous and isotropic and in such a state so as to continue expanding forever. In such a case, these physical aspects of our universe would be a temporal asymptotic consequence of Einstein's evolution equations, rather than having to be imposed externally as part of a cosmological principle, and thus would be a spectacular and dramatic cosmological confirmation of Einstein's equations.